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APPROXIMATE CONSTRUCTION OF SOLUTIONS IN GAME-THEORETIC CONTROL PROBLEMS†

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The problem of control using the feedback principle is considered, in which the aim is to ensure that a phase point falls within a terminal set no later than a specified time for any noises acting on the system, which are unknown in advance [1]. A method for the approximate construction of the set of positional absorption, that is, the set of all initial points for which the problem is solvable, is proposed. The relations defining the approximate set of positional absorption are stated. These relations differ from those proposed in [2] for the problem of approaching the terminal set at a given time. The results of an approximate computation of the set of positional absorption in the problem of controlling a pendulum in a viscous medium are presented. The paper touches on the topics considered in [1–16]. © 1997 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose that a conflict-control system is given whose behaviour in a time interval $[tg_0, \theta]$ ($t_0 < \theta < \infty$) is described by the equation

$$\dot{x} = f(t,x,u,v), \quad x[t_0]=x_0, \quad u \in P, \quad v \in Q \tag{1.1}$$

Here x is the m-dimensional phase vector of the system, u is the control function, v is the noise, and P and Q are compact sets in $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively.

It is assumed that (1.1) satisfies the standard conditions of game theory (see [1]). We consider the construction of the set of positional absorption W^0 consisting of all initial positions $(t_*, x_*) \in [t_0, \theta] \times \mathbf{R}^m$ for which there is a positional strategy $U(t, x)$ ensuring that $x(\tau) \in M$ for all $\tau \in [t_*, \theta]$ and for any $v(t)$ (the precise formulation of this problem is presented in [1]). The construction of W^0 in the problem under consideration is more difficult than the construction of the set of positional absorption in the problem in which the condition $x(\theta) \in M$ is to be satisfied.

The problem of approaching the target up to a fixed instant of time was studied in [1, 4–8], where, in particular, various constructions were considered suitable for analysing the general properties of W^0. Studies with a view to computations include a justification of the approximation of the differential equation by a difference equation.‡ Constructions related to an approximate computation of W^0 were considered in [9–11].

Below we will study problems related to the computational aspects of the construction of W^0. In Section 2 we define the stable absorption operator. The construction of the stable absorption operator is a quite general scheme serving as a basis for finding the positional absorption set W^0. In Section 3 we present conditions under which the discrete approximation of W^0 converges to W^0 as the discretization step tends to zero. We will present a relation which can be used to develop an algorithm for the approximate computation of W^0 for some classes of controlled systems in the plane. Examples are given in Section 4.

2. THE STABLE ABSORPTION OPERATOR

We assume that all the objects (stable bridges, motions or neighbourhoods of the target M) considered below are contained in a sufficiently large compact domain $D \subset [t_0, \theta] \times \mathbf{R}^m$.

We will consider the definition of a u-stable bridge, that is, the set of positions in which the motion can be preserved by a suitable choice of the control function u. The function (Hamiltonian)

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‡TARASYEV, A. M. and USHAKOV, V. N., On the construction of stable bridges in the minimax pursuit–evasion game. Sverdlovsk, 1983. Deposited at VINITI 19.04.83, No. 2454–83.

$$H(t,x,l)=\max_{u\in P}\min_{v\in Q}\langle l,f(t,x,u,v)\rangle,\quad l\in\mathbf{R}^m$$

of system (1.1) is in this definition. Here $\langle l,f\rangle$ denotes the scalar product of the vectors l and f.

Let $G=\{f\in\mathbf{R}^m\colon ||f||\leqslant K<\infty\}$ be a step such that $F(t,x)\subset G$ for any $(t,x)\in D$. Here $||f||=\langle f,f\rangle^{1/2}$.

Here a set Ψ of elements ψ is given along with a map $\{F_\psi\colon D\to 2^{\mathbf{R}^m}\}$ corresponding to Ψ and satisfying the following conditions.

A.1. For any $(t,x,\psi)\in D\times\Psi$ the set $F_\psi(t,x)$ is convex, closed and satisfies $F_\psi(t,x)\subset G$.

A.2. For any $(t,x,l)\in D\times S$

$$\min_{\psi\in\Psi}h_{F_\psi(t,x)}(l)=H(t,x,l).$$

A.3. There is a function $\omega^*(\delta)$ ($\omega^*(\delta)\downarrow 0$ as $\delta\downarrow 0$) such that

$$d(F_\psi(t^*,x^*),F_\psi(t_*,x_*))\leqslant\omega^*(|t^*-t_*|+\|x^*-x_*\|),$$

for any (t_*,x_*) and (t^*,x^*) from D and any D, $\psi\in\Psi$. Here $h_F(l)=\sup_{f\in F}\langle l,f\rangle$ for any $F\subset\mathbf{R}^m$, $S=\{l\in\mathbf{R}^m\colon ||l||=1\}$ and $d(F^*,F_*)$ is the Hausdorff distance between sets F^* and F_* in R^n.

As examples of families of maps satisfying conditions A.1–A.3 we consider the families $\{F_{v(\cdot)}\colon v(\cdot)\in V\}$ and $\{G_l\colon l\in S\}$ [12–15], where

$$G_l(t,x)=\{f\in G\colon\langle l,f\rangle\leqslant H(t,x,l)\},\quad F_{v(\cdot)}=\overline{\mathrm{co}}\{f(t,x,u,v(u))\colon u\in P\},$$

$\overline{\mathrm{co}}\{f\}$ is the closed convex hull of $\{f\}$ and V is the set of all functions $v(\cdot)\ P\to Q$.

Note that for some classes of controlled systems, in particular, for systems with right-hand side of the form

$$f(t,x,u,v)=\varphi(t,x)+B(t,x)u+C(t,x)v \tag{2.1}$$

and with P and Q being polyhedra with a finite number of vertices, one can introduce a family of maps satisfying A.1–A.3 such that Ψ is a finite set. This enables approximate constructions of W^0 to be realized at least for second-order systems.

Assuming $H\subset\mathbf{R}^m$, we introduce the following notation: $X_\psi(t^*;t_*,x_*)$ is the set of points $x^*\in\mathbf{R}^m$ reached at $t\in[t_*,t^*]$ by a solution $x(\cdot)=(x(\tau)t_*\leqslant\tau\leqslant t^*, x(t_*)=x_*$ of the differential inclusion $\dot{x}\in F_\psi(t,x)$

$$M_t(H)=\begin{cases}M, & t\in(t_*,t^*)\\ M\cup H, & t=t^*\end{cases}$$

$$X_\psi^{-1}(t_*;t,M_t(H))=(x_*\in\mathbf{R}^m\colon X_\psi(t^*,t_*,x_*)\cap M_t(H)\neq\emptyset\}$$

We will give a definition of the stable absorption operator in the problem of approaching the target M up to a fixed instant of time θ.

Definition 2.1. The map $\tau_1(t_*,t^*,\cdot)\colon 2^{\mathbf{R}^m}\to 2^{\mathbf{R}^m}$ given by

$$\pi(t_*;t^*,H)=\bigcap_{\psi\in\Psi}\bigcup_{t\in[t_*,t^*]}X_\psi^{-1}(t_*;t^*,M_t(H))$$

will be called the stable absorption operator $\pi(t_*;t^*,H)(t_0\leqslant t_*<t^*\leqslant\theta)$, $(H\subset\mathbf{R}^m)$ in the problem of approaching the target M up to time θ.

Definition 2.2. A closed set $W\subset D$ is called a minimax u-stable bridge in the problem of approaching M up to time θ if

$$W_\theta \subset M, \quad W_{t_*} \subset \pi(t_*; t^*, W_{t^*})$$

for any t_*, t^* ($t_0 \leqslant t_* < t^* \leqslant \theta$).

Here $W_t = \{x \in \mathbf{R}^m: (t,x) \in W\}$.

Let $\{F_\psi^{(i)}: D \to 2^{\mathbf{R}^m}\}$ ($i = 1, 2$) be two families of maps corresponding to $\Psi^{(i)}$ ($i = 1, 2$) and satisfying conditions A.1–A.3. Each of these families induces its own stable absorption operator $\pi^{(i)}(t_*; t^*, H)$. It can be shown that the operators are equivalent in the sense that a set $W \subset D$ which is a minimax u-stable bridge under one of the operators will be a minimax u-stable bridge under the other operator.

3. THE APPROXIMATING SYSTEM OF SETS AND ITS PROPERTIES

In addition to A.1–A.3 we shall assume that the family of maps $\{F_\psi: D \to 2^{\mathbf{R}^m}\}$ corresponding to Ψ satisfies the following condition .

A.4. There is a number $\lambda = \lambda(L) \in [0, \infty)$ such that

$$d(F_\psi(t,x^*),\ F_\psi(t,x_*)) \leqslant \lambda \| x^* - x_* \|, \quad \psi \in \Psi$$

for any (t, x_*) and (t, x^*) in D.

We denote by W^0 the maximal minimax u-stable bridge. We know that W^0 is the set of positional absorption in the problem under consideration.

We will define an approximating system of sets (ASS) with a view to an approximate computation of W^0. The notion of an ASS arises when the time-continuous scheme of u-stability is replaced by a discrete one, namely, when a division $\Gamma = \{t_0, t_1, \ldots, t_N = \theta\}$ is introduced and the domains $X_\psi(t^*; t_*, x_*)$ in Definition 2.2 are replaced by $x_* + (t^* - t_*)\, F_\psi(t_*, x_*)$.

Definition 3.1. By the approximating operator of stable absorption

$$a^\varepsilon(t_*; t^*, H)\,(\varepsilon \geqslant 0,\ t_0 \leqslant t_* < t^* \leqslant \theta,\ H \subset \mathbf{R}^m)$$

in the problem of approaching M up to time θ we shall mean the map $a^\varepsilon(t_*; t^*, \cdot): 2^{\mathbf{R}^m} \to 2^{\mathbf{R}^m}$ given by

$$a^\varepsilon(t_*; t^*, H) = \bigcap_{\psi \in \Psi} \bigcup_{t \in [t_*, t^*]} \tilde{X}_\psi^{-1}(t_*; t^*, M_t^\varepsilon(H))$$

Here

$$\tilde{X}_\psi^{-1}(t_*; t^*, M_t^\varepsilon(H)) = \{x_* \in \mathbf{R}^m: M_t^\varepsilon(H) \cap \tilde{X}_\psi(t; t_*, x_*) \neq \varnothing\}$$

$$\tilde{X}_\psi(t; t_*, x_*) = x_* + (t - t_*) F_\psi(t_*, x_*) \quad \text{for} \quad t \in [t_*, t^*].$$

$$M_t^\varepsilon(H) = \begin{cases} M_\varepsilon, & t \in [t_*, t^*) \\ M_\varepsilon \cup H, & t = t^* \end{cases}$$

M_ε being the ε-neighbourhood of M in $\mathbf{R}^m$.

We will say that a real-valued function $\eta(\Delta)(\Delta \geqslant 0)$ satisfies condition B if it is non-negative, decreases monotonically to zero as $\Delta \downarrow 0$, and $\lim_{\Delta \downarrow 0} \eta(\Delta)/\Delta = 0$.

Let $\Gamma_n = \{t_0, t_1, \ldots, t_{N(n)} = \theta\}$ be a given division of the interval $[t_0, \theta]$ and let $\eta(\Delta)$ be a function satisfying condition B.

We put

$$\eta^0(\Delta) \equiv 0, \quad \omega(\Delta) = \Delta\omega^*((1+K)\Delta)(\Delta \geqslant 0), \quad \Delta_i = t_{i+1} - t_i$$

$$\Delta^{(n)} = \max_{0 \leqslant i \leqslant N(n)-1} \Delta_i$$

$$\varepsilon_i = \varepsilon_i(\eta(\cdot)) = \omega(\Delta_{i-1}) + \eta(\Delta_{i-1}) + (1 + \lambda\Delta_{i-1})\varepsilon_{i-1}$$

$$\varepsilon_i^0 = \varepsilon_i(\eta^0(\cdot)) = \omega(\Delta_{i-1}) + (1+\lambda\Delta_{i-1})\varepsilon_{i-1}^0 \quad (i = 0,1,\ldots,N(n)-1)$$
$$\varepsilon_0 = \varepsilon_0^0 = 0$$

Definition 3.2. A system $\{{}_\eta\tilde{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$ given by the recursion relations

$${}_\eta\tilde{W}_{t_{N(n)}}^{(n)} = M_{\varepsilon_{N(n)}},\quad {}_\eta\tilde{W}_{t_i}^{(n)} = a^{\varepsilon_{i+1}}(t_i; t_{i+1}, {}_\eta\tilde{W}_{t_{i+1}}^{(n)})$$

$(i = N(n)-1, N(n)-2, \ldots, 0)$ will be called an η-ASS.

Let $\{\Gamma_n\}$ be an arbitrary sequence of divisions Γ_n of $[t_0, \theta\}$ such that $\lim_{n\to\infty} \Delta^{(n)} = 0$.

Definition 3.3. We denote by ${}_\eta W^0$ the set of all points (t_*, x_*) in D such that a sequence

$$\{(t_n, x_n): t_n = t_n(t_*) \in [t_0, \theta],\ x_n \in {}_\eta\tilde{W}_{t_n}^{(n)}, \lim_{n\to\infty} x_n = x_*\}$$

exists. Here

$$t_n(t_*) = \begin{cases} \min\limits_{(t_i\in\Gamma_n,\ t_i > t_*)} t_i, & t < \theta \\ t_*, & t = \theta \end{cases}$$

It has been shown that, subject to conditions similar to A.1–A.4, the sets W^0 and ${}_\eta W^0$ are the same for any function $\eta = \eta(\cdot)$ satisfying condition B. In a similar way it can be shown that W^0 and ${}_\eta W^0$ are identical for any function $\eta = \eta(\cdot)$ satisfying condition B, provided that conditions A.1–A.4 are satisfied.

The ASSs $\{{}_{\eta 0}\tilde{W}_{t_i}^{(n)} : t_i \in \Gamma_n\}$ and $\{{}_\eta\tilde{W}_{t_i}^{(n)} : t_i \in \Gamma_n\}$ corresponding to a division Γ_n and functions $\eta^0 = \eta^0(\cdot)$ and $\eta = \eta(\cdot)$ satisfy the inclusions

$${}_{\eta^0}\tilde{W}_{t_i}^{(n)} \subset {}_\eta\tilde{W}_{t_i}^{(n)},\quad t_i \in \Gamma_n$$

However, the η-ASS $\{{}_\eta\tilde{W}_{t_i}^{(n)} : t_i \in \Gamma_n\}$ introduced above is unsuitable for computations, since it is necessary to compute a non-denumerable number of sets

$$\tilde{X}_\psi^{-1}(t_i; t, M_t^{\varepsilon_i+1}({}_\eta\tilde{W}_{t_{i+1}}^{(n)})),\quad t \in [t_i, t_{i+1}]$$

in order to determine the set ${}_\eta\tilde{W}_{t_i}^{(n)}$ in $\mathbf{R}^m$.

We will consider the problem of constructing systems of set $\{{}_{\eta 0}\hat{W}_{t_i}^{(n)} : t_i \in \Gamma_n\}$ that can be computed efficiently enough and approximate W^0, that is, give W^0 in the limit as $n \to \infty$ ($\Delta^{(n)} \to 0$).

It has been shown that, for certain restrictions on the structure of M, one can introduce a system of sets $\{\hat{W}_{t_i}^n : t_i \in \Gamma_n\}$ that satisfy the relations

$${}_{\eta^0}\tilde{W}_{t_i}^{(n)} \subset \hat{W}_{t_i}^{(n)} \subset {}_\eta\tilde{W}_{t_i}^{(n)} \quad (i = 0,1,\ldots,N(n)) \tag{3.1}$$

and can be computed fairly easily, at least for problems in the plane. Namely, we consider the case when M can be represented as a union of spheres whose radii are bounded above by some $R^* \in (0, \infty)$. The system of sets $\{\hat{W}_{t_i}^n : t_i \in \Gamma_n\}$ is given by the recursion relations

$$\hat{W}_{t_{N(n)}}^{(n)} = M_{\varepsilon_{N(n)}},\quad \hat{W}_{t_i}^{(n)} = \bigcap_{\psi\in\Psi} \tilde{X}_\psi^{-1}(t_i; t_{i+1}, \hat{W}_{t_{i+1}}^{(n)} \cup M_{\varepsilon_{i+1}})$$

$(i = N(n)-1, \ldots, 1, 0)$.

Here $\varepsilon_{i+1} = \varepsilon_{i+1}((\eta(\cdot)), \eta(\Delta) = (K^2/R^*)\Delta^2, \Delta \geqslant 0$. From (3.1) and the equality

$$W^0 = \lim_{\substack{n\to\infty \\ \Delta^{(n)}\to 0}} \{{}_{\eta^0}\tilde{W}_{t_i}^{(n)}: t_i \in \Gamma_n\} = \lim_{\substack{n\to\infty \\ \Delta^{(n)}\to 0}} \{{}_\eta\tilde{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$$

it follows that

$$W^0 = \lim_{\substack{n\to\infty \\ \Delta^{(n)}\to 0}} \{\hat{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$$

where the limit is understood in the same sense as

$${}_\eta W^0 = \lim_{\substack{n\to\infty \\ \Delta^{(n)}\to\infty}} \{{}_\eta \tilde{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$$

i.e. in the sense of Definition 3.3.

In the case considered earlier $\hat{W}_{t_j}^{(n)}$ is the set of program absorption of the target $\hat{W}_{t_{i+1}} \cup M_{\varepsilon_{i+1}}$ at time t_{i+1} in the local (in time) approximation problem of approach in the time interval $[t_i, t_{i+1}]$. Approximation consists in that instead of the attainability domains $X_\psi(t_{i+1}: t_i, x[t_i])$ their linear (in time) approximations $\tilde{X}_\psi(t_{i+1}; t_i, x \mid t_i \mid)$ are considered. A similar approach to the construction of a system of sets approximating the resolvent set was applied in [6] when constructing a pursuit game. However, in the case when M is an arbitrary compact set, this method of approximating the resolving set W^0 in unsuitable.

We consider the case when M is an arbitrary compact set in $\mathbf{R}^m$. In this case to define the system of sets $\{\hat{W}_{t_i}^{(n)} : t_i \in \Gamma_n\}$ we fix a division Γ_n and an interval $[t_i, t_{i+1}]$ of the division Γ_n.

Assuming that $\hat{W}_{t_i}^{(n)}$ has already been constructed, we carry out a division $\Gamma_n^{(i)}$ of $[t_i, t_{i+1}]$ by instants of time $t_i^0 = t_i, t_i^1, \ldots, t_i^{N(n)} = t_{i+1}$ such that the diameter $\Delta_i^{(n)}$ of $\Gamma_n^{(i)}$ satisfies the equalities

$$\Delta_i^{(n)} = t_i^{k+1} - t_i^k = \frac{t_{i+1} - t_i}{N(n)}, \quad k = 0, 1, \ldots, N(n) - 1 \tag{3.2}$$

$N(n)$ being the number of intervals of Γ_n.

The inequalities

$$\Delta_i^{(n)} \leqslant \frac{(t_{i+1} - t_i)\Delta^{(n)}}{\theta - t_0}, \quad i = 0, 1, \ldots, N(n) - 1$$

follow from (3.2).

We introduce the function

$$\eta = \eta(\Delta) = \frac{K\Delta^{(n)}}{(\theta - t_0)}\Delta, \quad \Delta \geqslant 0 \tag{3.3}$$

and the set $\Psi \times \Gamma_n^{(i)} = \{(\psi, t_i^{(k)}): \psi \in \Psi, k = 0, 1, \ldots, N(n) - 1\}$.

Let us put

$$W_{t_i}^{\psi,k} = \begin{cases} \tilde{X}_\psi^{-1}(t_i, t_i^k, M_{\varepsilon_{i+1}}) & \text{for} \quad \psi \in \Psi, \quad k = 0, 1, \ldots, N(n) - 1 \\ \tilde{X}_\psi^{-1}(t_i, t_{i+1}, \hat{W}_{t_{i+1}}^{(n)} \cup M_{\varepsilon_{i+1}}) & \text{for} \quad \psi \in \Psi, \quad k = N(n) \end{cases}$$

and define the system of sets $\{\hat{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$ by the recursion relations

$$\hat{W}_{t_{N(n)}}^{(n)} = M_{\varepsilon_{N(n)}}, \quad \hat{W}_{t_i}^{(n)} = \bigcap_{\psi\in\Psi} \bigcup_{0 \leqslant k \leqslant N(n)-1} W_{t_i}^{\psi,k} \tag{3.4}$$

$i = N(n) - 1, \ldots, 1, 0.$

Theorem 3.1. The system of sets $\{\hat{W}_{t_i}^{(n)}: t_i \in \Gamma_n\}$ given by (3.4) satisfies (3.1) with $\eta(\cdot)$ given by (3.3).

Proof. Assuming that $\hat{W}_{t_i}^{(n)}$ are given by (3.4), we shall prove that

$$a^{\varepsilon_{i+1}^0}(t_i; t_{i+1}, \hat{W}_{t_{i+1}}^{(n)}) \subset \hat{W}_{t_i}^{(n)} \subset a^{\varepsilon_{i+1}}(t_i; t_{i+1}, \hat{W}_{t_{i+1}}^{(n)}) \tag{3.5}$$

Suppose that $x[t_i] \in a^{\varepsilon^0_{i+1}}(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}})$. Then by the definition of $a^{\varepsilon^0_{i+1}}(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}})$ we find that for any $\psi \in \Psi$ at least one of the two relations

$$\tilde{X}_\psi(t_{i+1}; t_i, x[t_i]) \cap \hat{W}^{(n)}_{t_{i+1}} \neq \emptyset, \quad \tilde{X}_\psi(\hat{t}; t_i, x[t_i]) \cap M_{\varepsilon^0_{i+1}} \neq \emptyset \tag{3.6}$$

is satisfied for some $\hat{t} \in [t_i, t_{i+1}]$.

We consider the case when the second condition in (3.6) is satisfied for $x[t_i]$. Assume that $[t_i^{\bar{k}}, t_i^{\bar{k}+1}]$ is the interval of Γ_n containing $\hat{t}$. By the second condition in (3.6) there is a vector $f_\psi \in F_\psi(t_i, x[t_i])$ such that

$$x[t_i] + (\hat{t} - t_i) f_\psi \in M_{\varepsilon^0_{i+1}} \tag{3.7}$$

Since $x[t_i] + (t - t_i)f_\psi$ satisfies (3.7)

$$\begin{aligned} x[t_i] + (t_i^{\bar{k}+1} - t_i) f_\psi &= (x[t_i] + (\hat{t} - t_i) f_\psi) + (t_i^{\bar{k}+1} - \hat{t}) f_\psi \in M_{(\varepsilon^0_{i+1} + K(t_i^{\bar{k}+1} - \hat{t}))} \subset \\ &\subset M_{(\varepsilon^0_{i+1} + K(t_i^{\bar{k}+1} - t_i^{\bar{k}}))} \subset M_{(\varepsilon^0_{i+1} + \eta(\Delta_i))} \end{aligned} \tag{3.8}$$

It can be seen that $\varepsilon^0_{i+1} + \eta(\Delta_i) \leqslant \varepsilon_{i+1} = \varepsilon_{i+1}(\eta(\cdot))$ for any i $(0 \leqslant i \leqslant N(n) - 1)$. It follows from (3.8) and the last inequality that $x[t_i] + (t_i^{\bar{k}+1} - t_i)f_\psi \in M_{\varepsilon+1}$, i.e. $\tilde{X}_\psi(\hat{t}_i^k; t_i, x[t_i]) \cap M_{\varepsilon i+1} \neq \varnothing$ $(k = \bar{k} + 1)$.

Consequently, if $x[t_i] \in a^{\varepsilon^0_{i+1}}(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}})$, then for any $\psi \in \Psi$ at least one of the two relations

$$x[t_i] \in \tilde{X}^{-1}_\psi(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}}), \quad x[t_i] \in \tilde{X}^{-1}_\psi(t_i; t_i^k, M_{\varepsilon_{i+1}})$$

holds for some k $(0 \leqslant k \leqslant N(n))$.

It follows that if $x[t_i] \in a^{\varepsilon^0_{i+1}}(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}})$, then $x[t_i] \in \hat{W}^{(n)}_{t_i}$.

On the other hand, $\hat{W}^{(n)}_{t_i} \subset a^{\varepsilon_{i+1}}(t_i; t_{i+1}, \hat{W}^{(n)}_{t_{i+1}})$ by the definition of $\hat{W}^{(n)}_{t_i}$. Inclusions (3.5) have therefore been proved.

Now we shall prove that

$$_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)-1}} \subset \hat{W}^{(n)}_{t_{N(n)-1}} \subset {}_\eta\tilde{W}^{(n)}_{t_{N(n)-1}} \tag{3.9}$$

Indeed, using the relations

$$_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)}} = \hat{W}^{(n)}_{t_{N(n)}} \subset {}_\eta\tilde{W}^{(n)}_{t_{N(n)}}$$

and (3.5), we obtain

$$\begin{aligned} {}_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)-1}} &= a^{\varepsilon^0_{N(n)}}(t_{N(n)-1}; t_{N(n)}, {}_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)}}) \subset a^{\varepsilon^0_{N(n)}}(t_{N(n)-1}; t_{N(n)}, \hat{W}^{(n)}_{t_{N(n)}}) \subset \hat{W}^{(n)}_{t_{N(n)-1}} \subset \\ &\subset a^{\varepsilon_{N(n)}}(t_{N(n)-1}; t_{N(n)}, \hat{W}^{(n)}_{t_{N(n)}}) \subset a^{\varepsilon_{N(n)}}(t_{N(n)-1}; t_{N(n)}, {}_\eta\tilde{W}^{(n)}_{t_{N(n)}}) = {}_\eta\overline{W}^{(n)}_{t_{N(n)-1}}. \end{aligned}$$

Inclusions (3.9) have thus been established.

Now we shall prove that

$$_{\eta^0}\tilde{W}^{(n)}_{t_{N(n-2)}} \subset \hat{W}^{(n)}_{t_{N(n)-2}} \subset {}_\eta\tilde{W}^{(n)}_{t_{N(n)-2}}. \tag{3.10}$$

Indeed, by (3.5) and (3.9) we obtain

$$\begin{aligned} {}_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)-2}} &= a^{\varepsilon^0_{N(n)-1}}(t_{N(n)-2}; t_{N(n)-1}, {}_{\eta^0}\tilde{W}^{(n)}_{t_{N(n)-1}}) \subset a^{\varepsilon^0_{N(n)-1}}(t_{N(n)-2}; t_{N(n)-1}, \hat{W}^{(n)}_{t_{N(n)-1}}) \subset \\ &\subset \hat{W}^{(n)}_{t_{N(n)-2}} \subset a^{\varepsilon_{N(n)-1}}(t_{N(n)-2}; t_{N(n)-1}, \hat{W}^{(n)}_{t_{N(n)-1}}) \subset a^{\varepsilon_{N(n)-1}}(t_{N(n)-2}; t_{N(n)-1}, {}_\eta\tilde{W}^{(n)}_{t_{N(n)-1}}) = \\ &= {}_\eta\tilde{W}^{(n)}_{t_{N(n)-2}}. \end{aligned}$$

Inclusions (3.10) have been verified.

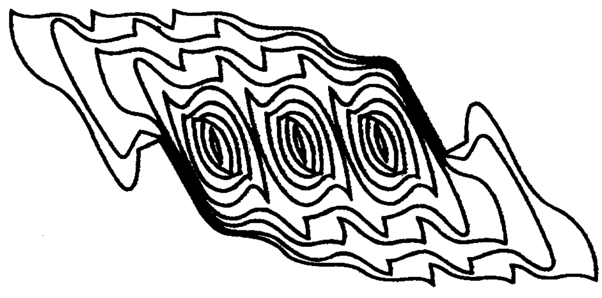

Fig. 1.

In a similar manner, by a recursive argument relations (3.1) can also be proved for $i = N(n)-3, \ldots, 1, 0$ in the case under consideration (for an arbitrary compact set M).

Note that if Ψ is a finite set, i.e. it has the form $\Psi = \{\psi^{\alpha}: \alpha = 1, 2, \ldots, \rho\}$, then the sets $\hat{W}_{t_i}^{(n)}$ are given by

$$\hat{W}_{t_i}^{(n)} = \bigcap_{1\le\alpha\le\rho} \; \bigcup_{0\le k\le N(n)-1} W_{t_i}^{\alpha,k}, \quad W_{t_i}^{\alpha,k} = W_{t_i}^{\psi_\alpha,k} \tag{3.11}$$

We can see from (3.1) that in this case one must compute $\rho \cdot N(n)$ sets $W_{t_i}^{\alpha,k}$ in order to compute $\hat{W}_{t_i}^{(n)}$.

4. APPROXIMATE COMPUTATION OF W_0 IN THE PROBLEM OF CONTROLLING A PLANE PENDULUM MOVING IN A VISCOUS MEDIUM

A number of studies [2, 16] have been devoted to the computational aspects of the solution of approach problems in various formulations. The most detailed computational scheme has been developed for solving approach problems in the case of linear controlled systems (1.1) [11, 16]. An algorithm for the approximate computation of the set of positional absorption W^0 in the problem of approaching the target at a fixed instant of time has been presented† in [2] in the case when (1.1) has the form (2.1). The basic elements of this algorithm are used to solve the problem considered here.

Consider a plane pendulum attached to a suspension point by a flexible unstretchable thread. The pendulum is controlled by an additional bounded force applied to it. The pendulum moves in a viscous medium, the parameters of which may vary with time. However, the exact value of the viscosity of the medium is unknown at any instant. Only the limits within which the viscosity can vary are known.

We assume that the equation of motion of the controlled plane pendulum has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -0.15 x_2 v - 10.15 \sin x_1 + u \tag{4.1}$$

Here the time interval in which the motion of pendulum is considered is [0. 2.25], $x = (x_1, x_2)$ is the phase space vector of system (4.1), u is the controlling moment such that $u \in [-10, 10]$, and v is the damping factor of the medium such that $v \in [0, 1]$.

The following problem can be formulated for system (4.1): it is required to damp the pendulum oscillations within a time interval not exceeding $\theta = 2.25$ or, which is the same, bring the phase vector $x = (x_1, x_2)$ of the system into the target set M consisting of one point (0, 0) no later than at the fixed time $\theta = 2.25$.

Note that, in addition to the equilibrium position (0, 0) system (4.1) has (in view of its periodicity) an infinite number of equilibria $(2k\pi, 0)$, where k is a natural number. Because of this the set of points $(2k\pi, 0)$ (k being a natural number) should be chosen as M. However, it is clear that for a target consisting of an infinite number of points it is impossible to construct the set of positional absorption using a computer. For this reason, we construct W^0 for M consisting of a finite number of points, for example, three points $(2k\pi, 0)$. For M consisting of three points $(-2\pi, 0)$ $(0, 0)$, $(2\pi, 0)$ the sets $\tilde{W}(t_i)$ in the interval [0. 2.25] are shown in Fig. 1. They are separated from one another in time by 0.25.

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